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Orthogonality Properties of Transverse Eigenmodes of Phase Conjugate Optical Resonators

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Abstract—**The orthogonality properties of the transverse eigenmodes of optical resonators which have phase conjugate mirrors at both ends are derived. As in conventional resonators and also resonators with only one phase conjugate mirror, it is shown that the transverse eigenmodes are essentially biorthogonal, a relation which is satisfied between the set of modes propagating in one direction around the resonator and the adjoint set of modes propagating in the reverse direction.**

I. Introduction

One of the main functions of phase-conjugate mirrors (PCM's) is to generate electromagnetic waves which retrace the original path of incident waves, with phase distributions which are the exact opposite of those of the incoming waves [1]. The reflected wave is, therefore, a time-reversed replica of the incident field.

Thus, it has been suggested that when a PCM is used to replace one of the two conventional mirrors that form a laser cavity, it may also compensate for phase distortions contributed by poor intracavity optical components [2]. Whereas a conventional resonator is stable over a certain range of mirror curvatures and separations, the phase-conjugate resonator (PCR), due to the time-reversal properties of the conjugate wave, is stable regardless of the cavity spacing and/or the conventional mirror's curvature [3]. The PCR is the least sensitive to internal phase fluctuations, corrects almost perfectly aberrations located near the PCM, and is the resonator with the smallest diffraction losses for a given dimension [4]. Moreover the resonance frequencies are not sensitive to changes in the length of the resonator. Because of these properties, the use of PCR's to attain maximum output power in the fundamental TEM_{00} mode seems promising [4].

It is known that complex paraxial elements such as Gaussian or hard-edged apertures must be added to the resonator to obtain a unique mode such as the TEM_{00} mode [5]. When the resonator contains hard-edged apertures or Gaussian apertures, numerical procedures are necessary to derive the spatial distributions of the mode. Most of these procedures are based on the Fox–Li type calculation [6]. Knowledge of the orthogonality properties of the transverse modes, however, can be most helpful for the computations involved. For these reasons, Siegman and Hardy have verified the orthogonality relation for the conventional resonator and the PCR with a PCM at one end, respectively [7], [8]. In this letter, we drive for the first time the orthogonality relations of the eigenmodes of the PCR with PCM's at each end.

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II. Orthogonality Properties of Transverse Eigenmodes in a PCR

Let us consider an optical resonator which has PCM's at both ends as shown in Fig. 1. The cavity contains an arbitrary set of paraxial optical elements represented by an *ABCD* matrix. In the following discussion, we assume that the reflection from the PCM is accomplished by means of degenerate four-wave mixing (DFWM). If the nonlinear interaction in the two PCM's is given by nondegenerate four-wave mixing, after several round trips, the frequency of the probe would walk off the gain spectrum of both PCM's. This means that oscillations between the two PCM's are not possible unless the mirrors are pumped by the lasers of the same frequency. We also assume that the reflection from the phase conjugator PCM2 may have an arbitrary transverse variation in magnitude but not in phase as described by

$$E_r(x) = \rho_{\mathrm{pcm}}(x)\, E_i^*(x) = \left|\rho_{\mathrm{pcm}}(x)\right| \exp(j\psi_x)\, E_i^*(x) \tag{1}$$

where $E_i(x)$ is the incident field on PCM2, $E_r(x)$ is the phase conjugated reflected field of the PCM2, $|\rho_{\mathrm{pcm}}(x)|$ is the gain or loss of the PCM2, and, whether PCM2 is apertured or ideal, the phase ψ_x is not a function of the transverse coordinate x and y. Similarly, we assume the reflection from the PCM1 at the other end of the resonator may have an arbitrary transverse variation in magnitude but not in phase as described by

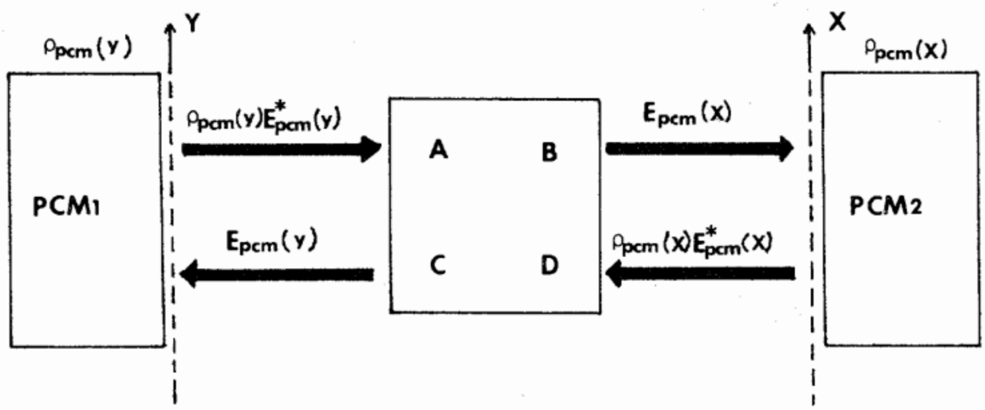

Fig. 1. Schematic diagram of an optical resonator with PCM's at each end.

$$E_r(y) = \rho_{\text{pcm}}(y)\, E_i^*(y) = |\rho_{\text{pcm}}(y)| \exp(j\psi_y)\, E_i^*(y) \tag{2}$$

where $E_i(y)$ is the incident field on PCM1, $E_r(y)$ is the phase conjugated reflected field of the PCM1, $|\rho_{\text{pcm}}(y)|$ is the loss or gain of the PCM1, and, whether PCM1 is apertured or ideal, the phase ψ_y is not a function of the transverse coordinate x and y.

Let us assume that a transverse eigenmode E_{pcm} is incident on PCM2 as shown in Fig. 1. The resulting field distribution $E_{\text{pcm}}(y)$ just in front of PCM1 can then be written in terms of a generalized Huygens integral [9] as

$$\gamma_1 E_{\text{pcm}}(y_1) = \left(\frac{jk}{2\pi B}\right)^{1/2} \int \rho_{\text{pcm}}(x_1)\, E_{\text{pcm}}^*(x_1) \cdot \exp\left[-\frac{jk}{2B}\left(Ax_1^2 - 2x_1y_1 + Dy_1^2\right)\right] dx_1 \tag{3}$$

where γ_1 is a complex constant representing the axial phase shift, x_1 is the variable of PCM2, y_1 is that of PCM1 during one round trip in the resonator, and k is the wavenumber. The field distribution $E_{\text{pcm}}(x)$ in front of the PCM2 can be likewise written in terms of $E_{\text{pcm}}(y)$ as

$$\gamma_2 E_{\text{pcm}}(x_2) = \left(\frac{jk}{2\pi B}\right)^{1/2} \int \rho_{\text{pcm}}(y_1)\, E_{\text{pcm}}^*(y_1) \cdot \exp\left[-\frac{jk}{2B}\left(Dy_1^2 - 2y_1x_2 + Ax_2^2\right)\right] dy_1 \tag{4}$$

where the complex matrix $DBCA$ for propagation in reverse direction was used, γ_2 is a complex constant representing the axial phase shift, and x_2 is the variable of PCM2 after one round trip around the resonator. By use of (3) and (4), we obtain an eigenequation for E_{pcm}, namely

$$\gamma_1^*\gamma_2^2 E_{\text{pcm}}(x_2) = \left|\frac{k}{2\pi B}\right| \iint \rho_{\text{pcm}}(y_1)\, \rho_{\text{pcm}}^*(x_1)\, E_{\text{pcm}}(x_1) \cdot \exp\left[\frac{jk}{2}\left(\frac{A^*}{B^*}x_1^2 - \frac{2x_1y_1}{B^*} + \frac{D^*}{B^*}y_1^2 - \frac{D}{B}y_1^2 + \frac{2y_1x_2}{B} - \frac{A}{B}x_2^2\right)\right] dx_1\, dy_1. \tag{5}$$

Equation (5) describes the resulting field distribution just in front of PCM2 after one complete round trip around the resonator. As we shall see later, when we investigate whether or not the kernel of the eigenequation is Hermitian, it is noted that when variables x_1 and x_3 are interchanged, one must also interchange the dummy variables y_1 and y_2 simultaneously, where y_2 is the variable of PCM1 during the second round trip and x_3 is that of PCM2 after the second round trip. We have need of an equation which contains the variable y_2. That is, because y_2 is the variable of PCM1 during the second round trip, we must obtain the resulting field distribution in front of PCM2 after the second round trip in the resonator.

After the transverse eigenmode E_{pcm} has made the second round trip in the resonator, the resulting field distributions in front of PCM1 and PCM2 are given as follows

$$\gamma_1 E_{\text{pcm}}(y_2) = \left(\frac{jk}{2\pi B}\right)^{1/2} \int \rho_{\text{pcm}}(x_2)\, E_{\text{pcm}}^*(x_2) \cdot \exp\left[-\frac{jk}{2B}\left(Ax_2^2 - 2x_2y_2 + Dy_2^2\right)\right] dx_2 \tag{6}$$

$$\gamma_2 E_{\text{pcm}}(x_3) = \left(\frac{jk}{2\pi B}\right)^{1/2} \int \rho_{\text{pcm}}(y_2)\, E_{\text{pcm}}^*(y_2) \cdot \exp\left[-\frac{jk}{2B}\left(Dy_2^2 - 2x_3y_2 + Ax_3^2\right)\right] dy_2. \tag{7}$$

We can obtain, by use of (6) and (7) in a procedure similar to the one resulting in (5), an eigenequation representing the resulting field distribution in front of PCM2 after the round trip around the resonator. It is given by

$$\gamma_1^* \gamma_2 E_{\text{pcm}}(x_3) = \left|\frac{k}{2\pi B}\right| \iint \rho_{\text{pcm}}(y_2)\, \rho_{\text{pcm}}^*(x_2)\, E_{\text{pcm}}(x_2) \cdot \exp\left[\frac{jk}{2}\left(\frac{A^*}{B^*}x_2^2 - \frac{2x_2y_2}{B^*} + \frac{D^*}{B^*}y_2^2 - \frac{D}{B}y_2^2 + \frac{2y_2x_3}{B} - \frac{A}{B}x_3^2\right)\right] dx_2\, dy_2. \tag{8}$$

Then by using (5) and (8), we can obtain the resulting transverse field distribution just in front of PCM2 after two complete round trips around the resonator. From this eigenequation we can derive the eigenequation which is needed to investigate the orthogonality relations between the set of transverse eigenmodes incident on the PCM and the set of the reflected transverse eigenmodes. Such a relation is described by

$$\begin{aligned}(\gamma_1^*\gamma_2)^2 E_{\text{pcm}}(x_3) &= \left|\frac{k}{2\pi B}\right|^2 \iiiint \rho_{\text{pcm}}^*(x_1)\, \rho_{\text{pcm}}^*(x_2) \\ &\cdot \rho_{\text{pcm}}(y_1)\, \rho_{\text{pcm}}(y_2)\, E_{\text{pcm}}(x_1) \\ &\cdot \exp\left[k\left\{\text{Im}\left(\frac{A}{B}\right)x_2^2 + \text{Im}\left(\frac{D}{B}\right)(y_1^2 + y_2^2)\right\}\right] \\ &\cdot \exp\left[-\frac{jk}{2}\left\{\frac{A}{B}x_3^2 - \left(\frac{A}{B}\right)^* x_1^2 - \frac{2}{B}y_2x_3 + \left(\frac{2}{B}\right)^* y_1x_1 \right.\right. \\ &\left.\left. - \frac{2}{B}y_1x_2 + \left(\frac{2}{B}\right)^* x_2y_2\right\}\right] dx_1\, dy_1\, dx_2\, dy_2\end{aligned} \tag{9}$$

where Im (A/B) and Im (D/B) are the imaginary parts of A/B and D/B, respectively. If we assume that the structure and properties of the two PCM's are identical, then we can obtain the relation $\psi_x = \psi_y$. Therefore, by use of (1), (2), and (9), we can obtain the representation

$$\rho_{\text{pcm}}^*(x_1)\, \rho_{\text{pcm}}^*(x_2)\, \rho_{\text{pcm}}(y_1)\, \rho_{\text{pcm}}(y_2) = |\rho_{\text{pcm}}(x_1)|\, |\rho_{\text{pcm}}(x_2)|\, |\rho_{\text{pcm}}(y_1)|\, |\rho_{\text{pcm}}(y_2)|. \tag{10}$$

Then, by use of (9) and (10), we obtain the following eigenequation:

$$(\gamma_1^*\gamma_2)^2 E_{\text{pcm}}(x_3) = \left|\frac{k}{2\pi B}\right|^2 \int |\rho_{\text{pcm}}(x_1)| \cdot E_{\text{pcm}}(x_1)\, G(x_3, x_1)\, dx_1 \tag{11}$$

where the kernel $G(x_3, x_1)$ is given by

$$\begin{aligned}G(x_3, x_1) &= \iiint |\rho_{\text{pcm}}(x_2)|\, |\rho_{\text{pcm}}(y_1)|\, |\rho_{\text{pcm}}(y_2)| \\ &\cdot \exp\left[k\left\{\text{Im}\left(\frac{A}{B}\right)x_2^2 + \text{Im}\left(\frac{D}{B}\right)(y_1^2 + y_2^2)\right\}\right] \\ &\cdot \exp\left[-\frac{jk}{2}\left\{\left(\frac{A}{B}x_3^2 - \left(\frac{A}{B}\right)^* x_1^2 - \frac{2}{B}y_2x_3 \right.\right.\right. \\ &+ \left(\frac{2}{B}\right)^* y_1x_1 - \frac{2}{B}y_1x_2 \\ &\left.\left. + \left(\frac{2}{B}\right)^* x_2y_2\right\}\right] dy_1\, dx_2\, dy_2.\end{aligned} \tag{12}$$

Equation (12) is the desired equation which is needed to verify the orthogonality of the transverse eigenmodes of a PCR with PCM's at both ends. Now let define the modified eigenfunction and kernel

$$u(x) = \rho_{\text{pcm}}^{1/2}(x)\, E_{\text{pcm}}(x) = |\rho_{\text{pcm}}(x)|^{1/2} \exp\left(j\frac{\psi_x}{2}\right) E_{\text{pcm}}(x). \tag{13}$$

$$v(x_3, x_1) = |\rho_{\text{pcm}}(x_1)|^{1/2}\, |\rho_{\text{pcm}}(x_3)|^{1/2}\, G(x_3, x_1). \tag{14}$$

By substituting (13) and (14) into (11), the eigenequation (11) is reduced to the form

$$\alpha u(x_3) = \int u(x_1)\, v(x_3, x_1)\, dx_1 \tag{15}$$

where

$$\alpha = \left\{\left|\frac{2\pi B}{k}\right| \gamma_1^*\gamma_2\right\}^2.$$

It is now easily verified by means of (12) and (14) that the kernel $v(x_3, x_1)$ is Hermitian; that is, it satisfies

$$v(x_3, x_1) = v^*(x_1, x_3). \tag{16}$$

When variables x_1 and x_3 are interchanged, one must also interchange variables y_1 and y_2 for PCM1 to obtain (16). Since the kernel $v(x_3, x_1)$ of (15) is Hermitian, its eigenfunctions $u(x)$ form a complete set of orthonomal functions [10] in the sense that

$$\int u_n^*(x)\, u_m(x)\, dx = \delta_{nm}. \tag{17}$$

By introducing (13) and (1) into (17), and after proper normalization, one obtains the following representation which describes the orthogonality relations between the transverse eigenmodes of the resonator.

$$\int E_{f,m} E_{b,n}\, dx = \delta_{nm}. \tag{18}$$

This orthogonality relationship between any forward traveling transverse eigenmode E_f and any backward traveling transverse eigenmode E_b is satisfied for all planes perpendicular to the optical axis of the resonator. Equation (18) is essentially the same as orthogonality relation obtained by Siegman for conventional optical resonators and by Hardy *et al.* for the PCR which has one PCM at one end [11].

III. Conclusion

We have derived the relationship for the orthogonality of the transverse eigenmodes of PCR's terminated by identical PCM's which are of the degenerate four-wave mixing sort. As in conventional resonators and in optical resonators which have only one PCM, we have shown that the eigenmodes which propagate in one direction are orthogonal to the modes which propagate in the opposite direction over any plane perpendicular to the optical axis of the resonator.

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